

Conditions for entanglement transformation between a class of multipartite pure states with generalized Schmidt decompositions

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(Dated: February 1, 2008)

We generalize Nielsen's majorization criterion for the convertibility of bipartite pure states [Phys. Rev. Lett **83**, 436(1999)] to a special class of multipartite pure states with generalized Schmidt decompositions.

PACS numbers: 03.67.-a, 03.65.Ud

One of the central problems of quantum entanglement is to find conditions under which an entangled state can be transformed into another one by local operations and classical communication [1]. In 1999 Nielsen reported a sufficient and necessary condition for the deterministic entanglement transformations between bipartite pure states [2]. Nielsen's work has been extended to the case when a deterministic transformation cannot be achieved [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. These efforts indicate that the structure of bipartite pure entanglement has been well understood.

But all the works above are only on bipartite pure states and very little is known about the structure of multipartite pure states [20]. Recently the study of entanglement transformation between multipartite states has received considerable attentions [21, 22]. It is of great interest to generalize Nielsen's result to a multipartite scenario. Such a result should lead to a deep understanding of the nature of multipartite entanglement. For instance, it can be used to identify what kind of multipartite entangled states are universal resources in realizing one-way quantum computation [23].

In this Brief Report, we consider entanglement transformations of a class of multipartite pure states which have generalized Schmidt decompositions. We show that Nielsen's theorem can be extended to this class of states. Furthermore, our result confirms the intuition that the entanglement of a multipartite state with a generalized Schmidt decomposition is completely determined by its Schmidt coefficient vector.

Suppose Alice, Bob, ..., and Dana share a multipartite pure state $|\psi\rangle$ which has a generalized Schmidt decomposition as follows:

$$|\psi\rangle = \sum_{k=0}^{n-1} \sqrt{\lambda_k} |k\rangle_A |k\rangle_B \cdots |k\rangle_D, \quad (1)$$

where $\{|k\rangle\}_A$, $\{|k\rangle\}_B$, ..., and $\{|k\rangle\}_D$ are orthonormal bases for Alice, Bob, ..., and Dana respectively, and $\lambda =$

$(\lambda_0, \dots, \lambda_{n-1})$ represents the Schmidt coefficient vector with nonincreasing order— i.e., $\lambda_0 \geq \dots \geq \lambda_{n-1} \geq 0$. They want to transform $|\psi\rangle$ to the following state $|\phi\rangle$ using LOCC:

$$|\phi\rangle = \sum_{k=0}^{n-1} \sqrt{\mu_k} |k'\rangle_A |k'\rangle_B \cdots |k'\rangle_D, \quad (2)$$

where $\{|k'\rangle\}_A$, $\{|k'\rangle\}_B$, ..., and $\{|k'\rangle\}_D$ are also orthonormal bases for Alice, Bob, ..., and Dana, respectively, and $\mu = (\mu_0, \dots, \mu_{n-1})$ is the Schmidt coefficients vector with non-increasing order. Two sets of bases $\{|k\rangle\}$ and $\{|k'\rangle\}$ are generally different. We say λ is majorized by μ , denoted as $\lambda \prec \mu$, if

$$\sum_{k=0}^l \lambda_k \leq \sum_{k=0}^l \mu_k, \quad 0 \leq l \leq n-2,$$

and $\sum_{k=0}^{n-1} \lambda_k = \sum_{k=0}^{n-1} \mu_k$. With these notations, we have the following generalization of Nielsen's theorem.

Theorem 1. Alice, Bob, ..., Dana can transform $|\psi\rangle$ to $|\phi\rangle$ using LOCC if and only if $\lambda \prec \mu$.

When there are only two parties, the above theorem is exactly the one by Nielsen [2]. This is due to the fact that any bipartite pure state has a Schmidt decomposition. In general, there exist multipartite states not having such a decomposition [24]. So the above theorem only covers a special class of multipartite states. We would like to point out that a characterization of the existence of a generalized Schmidt decomposition has already been given by Thapliyal [25].

Since we can split all these parties into two groups and treat each group altogether as a single party, the necessity of the condition $\lambda \prec \mu$ follows directly from Nielsen's theorem. However, whether this condition is also sufficient for the convertibility remains unknown. Actually, the protocol given by Nielsen consists of a local measurement by one party and conditional unitary operations by the other party. When there are more than two parties, it is not clear whether all these operations can be implemented locally. Fortunately, we notice that in Ref. [27]

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Jensen and Schack have presented an alternative protocol for entanglement transformations between bipartite pure states, which simplifies the Nielsen's original proof considerably. Although they are only concerned with bipartite pure states, we shall show that with minor modifications their protocol can be used to prove that majorization is also a sufficient condition for entanglement transformations between multipartite states with generalized Schmidt decompositions.

We shall employ a useful alternative characterization of majorization [26]. That is, if $\lambda \prec \mu$, then we can write $\lambda = \sum_{j=1}^N p_j \sigma_j \mu$ for a probability distribution $\{p_j\}$ and a set of permutations $\{\sigma_j\}$, where N is at most $(n-1)^2 + 1$. More explicitly, we have

$$\sum_{j=1}^N p_j \mu_{\sigma_j^{-1}(k)} = \lambda_k, \quad k = 0, \dots, n-1. \quad (3)$$

Let

$$M_j^A = \sqrt{p_j} \sum_{k=0}^{n-1} \sqrt{\frac{\mu_{\sigma_j^{-1}(k)}}{\lambda_k}} |k_A\rangle \langle k_A|, \quad j = 1, \dots, N. \quad (4)$$

It follows from Eq. (3) that $\sum_{j=1}^N M_j^{A\dagger} M_j^A = I^A$. So $\{M_j^A\}$ is a complete quantum measurement. A simple local protocol that transforms $|\psi\rangle$ to $|\phi\rangle$ consists of the following two steps.

Step 1. Alice performs $\{M_j^A\}$ on her subsystem, then broadcasts the measurement outcome j to other parties;

Step 2. Every party performs a unitary operation U_j on his/her own subsystem if the measurement outcome is j , where $U_j = \sum_{k=0}^{n-1} |\sigma_j^{-1}(k')\rangle \langle k|$.

The validity of the above protocol can be verified as follows. By Eqs. (3) and (4), Alice obtains the measurement outcome j with probability p_j , and the post-measurement state is changed into

$$|\psi_j\rangle = \sum_{k=0}^{n-1} \sqrt{\mu_{\sigma_j^{-1}(k)}} |k\rangle_A |k\rangle_B \cdots |k\rangle_D. \quad (5)$$

Then every party then performs a unitary operation U_j on his/her own subsystem. After that, $|\psi_j\rangle$ is transformed into

$$|\phi_j\rangle = \sum_{k=0}^{n-1} \sqrt{\mu_{\sigma_j^{-1}(k)}} |\sigma_j^{-1}(k')\rangle_A |\sigma_j^{-1}(k')\rangle_B \cdots |\sigma_j^{-1}(k')\rangle_D. \quad (6)$$

Relabeling the subscript $\sigma_j^{-1}(k)$ as k , we can see that the final output state $|\phi_j\rangle$ is exactly $|\phi\rangle$.

As an illustrative example, let us consider the special case when $n = 2$. Without loss of generality, we can assume $|0'\rangle = |0\rangle$ and $|1'\rangle = |1\rangle$. In this case there are only two permutations I and X . Then theorem 1 indicates that $|\psi\rangle$ can be transformed to $|\phi\rangle$ if and only if $\lambda_1 \leq \mu_1$. We shall describe a simple protocol for the transformation. If $\lambda_1 = \mu_1$, then $|\psi\rangle$ and $|\phi\rangle$ are equivalent up to

some local unitary. Assume that $\lambda_1 < \mu_1$. We can choose $0 < p < 1$ such that

$$[pI + (1-p)X]\mu = \lambda. \quad (7)$$

A simple calculation indicates that

$$p = \frac{\lambda_1 - \mu_2}{\mu_1 - \mu_2}.$$

Then Alice performs a measurement $\{M_0, M_1\}$ on her subsystem, where

$$M_0 = \sqrt{p} \left(\sqrt{\frac{\mu_0}{\lambda_0}} |0\rangle \langle 0| + \sqrt{\frac{\mu_1}{\lambda_1}} |1\rangle \langle 1| \right) \quad (8)$$

and

$$M_1 = \sqrt{1-p} \left(\sqrt{\frac{\mu_1}{\lambda_0}} |0\rangle \langle 0| + \sqrt{\frac{\mu_0}{\lambda_1}} |1\rangle \langle 1| \right). \quad (9)$$

If the measurement outcome is 0, then the final output state is already $|\phi\rangle$ and nothing needs to be done. Otherwise, every party performs a bit-flip operation X on their subsystems.

We would like to point out that many results valid for bipartite pure states can be directly generalized to multipartite pure states with generalized Schmidt decompositions, as these results only depend on Schmidt coefficient vectors [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. For instance, using the very same method, we can easily show the following generalized Vidal's formula [6] for a multipartite quantum system.

Theorem 2. The maximal conversion probability from $|\psi\rangle$ to $|\phi\rangle$ using LOCC is given by

$$p_{\max}(\psi, \phi) = \min \left\{ \frac{E_l(\psi)}{E_l(\phi)} : 0 \leq l \leq n-1 \right\}, \quad (10)$$

where $E_l(\psi) = \sum_{k=l}^{n-1} \lambda_k$.

The right-hand side of Eq. (10) is an upper bound for p_{\max} and follows directly from the optimality of Vidal's result for bipartite pure states, as we can always split all parties into two groups as we did in the proof of theorem 1. We only need to show that there really exists a local protocol which can attain this upper bound. Our protocol is almost the same as the one given by Vidal [6]. We first convert $|\psi\rangle$ into a temporary state $|\Omega\rangle$ with certainty, then further convert $|\Omega\rangle$ into $|\phi\rangle$ by performing a local measurement.

More precisely, suppose $\gamma = (\gamma_0, \dots, \gamma_{n-1})$ is the Schmidt coefficient vector of $|\Omega\rangle$ which is constructed using the same method as in Ref. [6]. From the construction of $|\Omega\rangle$, we obtain that $\lambda \prec \gamma$. Applying theorem 1, we can make sure the conversion from $|\psi\rangle$ to $|\Omega\rangle$ can be done using LOCC with certainty. So the first step in Vidal's proof to convert $|\psi\rangle$ into $|\phi\rangle$ can be achieved locally. The proof that $|\Omega\rangle$ can be locally transformed into

$|\phi\rangle$ with probability p_{\max} is exactly the same as that in Ref. [6].

As another instance, one can easily see that surprising phenomena such as entanglement catalysis [5], partial recovery of entanglement [8], multiple-copy entanglement transformation [9], and mutual catalysis [10] also exist in multipartite quantum systems, and many interesting properties concerning with these phenomena are still valid for multipartite pure states with generalized Schmidt decompositions [11, 12, 13, 14, 15, 16, 17, 18].

We are indebted to the colleagues in the Quantum

Computation and Quantum Information Research Group for many enjoyable conversations. In particular, we sincerely thank Professor Mingsheng Ying for his numerous encouragement and constant support of this research, and one of us (R. D.) wishes to thank Professor Yuan Feng for countless inspiring discussions on entanglement transformation. This work was partly supported by the National Natural Science Foundation of China (Grants Nos. 60621062 and 60503001) and the Hi-Tech Research and Development Program of China (863 project) (Grant No. 2006AA01Z102).

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